Some Types of Compactness via Ideal

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Abstract— In the present paper, we introduce and study some types of compactness modulo an ideal called βI-compact and countably βI-compact spaces. These concepts generalize β-compactness and β-Lindelofness .Several effects of various types of functions on them are studied.

Index Terms— Topological ideal, I-compact, countably I-compact, β-compact, βI-compact and countably βI-compact
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1 Introduction

One of the most powerful notions in system analysis is the concept of topological structures and their generalizations. In principal, generalized open sets play a crucial role in general topology, they are interesting research topics of many topologists. Moreover, generalized open sets are of significant applications in general topology and real analysis, which concern various modified forms of continuity, separation axioms etc. Recently, the generalized open sets (α, β, semi-, pre- and γ) open sets are utilized to develop the rough sets and approximation spaces [2]; which could be extended to computer science and information systems. Our goal in this work is to use the concepts of compactness and ideals to enrich topological spaces with new tools that help in applications. Levine [8] introduced the concept of semi-open sets. In 1983 Abd El-Monsef et al. [1] introduced the concept of β-open sets in topological spaces. The concept of compactness modulo an ideal was first defined by Newcomb [12] and has been studied by Rancin [13]. In 1990 it has been extensively studied by Hamlett and Jankovic [6]. Newcomb [12] also defined the concept of countable compactness modulo an ideal.

The purpose of the present paper is to introduce and study some types of compactness modulo an ideal called βI-compact and countably βI-compact spaces.

Throughout we work with a topological space (X, τ) (or simply X), where no separation axioms are assumed. The usual notation Cl(A) for the closure and Int(A) for the interior of a subset A of a topological space (X, τ).

2 Preliminaries

Definition 2.1 [7] A non-empty collection I of subsets of a non-empty set X is said to be an ideal on X, if it satisfies:
(a) A∈I and B⊆A → B∈I.
(b) A∈I and B∈I→ A∪B∈I.

We denote by I_f (resp., I_c) the ideal of finite (resp., countable) subsets of X.

Lemma 2.2 [6] (a) The intersection of two ideals on a non-empty set X is an ideal, but the union of two ideals is not an ideal in general.
(b) The sum I∨J of two ideals I and J on a non-empty set X is the ideal {E∪H : E∈I and H∈J}.

Definition 2.3 A subset A of a topological space (X,τ) is said to be:
(i) β-open [1](semi-pre-open [4]), if A⊆Cl Int Cl A.
(ii) semi-open [8], if A⊆Cl Int A.
(iii) γ-open [10], if A⊆Int Cl A∪Cl Int A.

The complement of β-open (resp., semi-open, γ -open) is β-closed (resp., semiclosed, γ-closed).

The class of all β-open (resp., semi open, γ-open) sets of topological space (X,τ) are denoted by βO(X, τ) (resp., SO(X, τ), γO(X, τ)). Also, τ ⊆SO(X,τ) ⊆γO(X,τ)⊆βO(X, τ).

Definition 2.4 A topological space (X,τ) is said to be:
(a) Submaximal [5], if every dense subset of X is open in X.
(b) Extremlly disconnected (briey E.D.) [15], if the closure of each open set of X is open in X.

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Theorem 2.5 If (X,τ) is E.D., then β-open set is γ-open set.

Definition 2.6 A topological space (X,τ) is said to be:

(a) β-compact [9], if every β-open cover of X has a finite subcover.

(b) Countably β-compact [9], if every countable β-open cover of X has a finite subcover.

(c) β-Lindelof [8], if every β-open cover of X has a countable subcover.

Definition 2.7 A function $f:(X,\tau)\rightarrow(Y,\sigma)$ is said to be:

(a) β-irresolute [9], if the inverse image of each β-open set in Y is a β-open set of X.

(b) β-continuous [1], if the inverse image of each open set in Y is a β-open set of X.

(c) Mβ-open [11], if the image of each β-open set in X is a β-open set of Y.

Given a topological space (X,τ) and $x\in X$, we define by $\tau(x) = \{U\in\tau : x\in U\}$. Let (X, τ, I) be a topological space with ideal I and $A\subseteq X$, then the local function of A with respect to I and τ [14] $A^*(I)= \{x\in X: U\cap A\notin I$ for each $U\in\tau(x)\}$. For every topological space (X,τ,I) with ideal I, there exists a topology $\tau^*(I)$, finer than τ, generated by the base $\mathfrak{b}(I,\tau)=\{U-E : U\in\tau \text{ and } E\in I\}$. Additionally, $Cl^*(A)=A\cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$.

Lemma 2.8 [12] The following properties hold:

(a) If $f:(X, \tau, I)\rightarrow(Y, \sigma)$ is a surjection function, then $f(I) =\{f(A) : A\in I\}$ is an ideal on Y.

(b) If $f:(X, \tau)\rightarrow(Y, \sigma, J)$ is an injection function, then $f^{-1}(J) =\{f^{-1}(B) :B\in J\}$ is an ideal on X.

Definition 2.9 A subset A of a space (X, τ, I) is said to be I-compact [12] (resp., SI-compact [3]), if for every cover $\{U_\lambda: \lambda\in\Lambda\}$ of A by open (resp., semi-open) sets of X, there exists a finite subset Λ_0 of Λ such that $A-\cup\{U_\lambda : \lambda\in\Lambda_0\}\in I$. The space (X, τ, I) is said to be I-compact (resp., SI-compact), if X is I-compact (resp., SI-compact) as a subset

3 βI-COMPACT SPACES

Definition 3.1 A space (X, τ, I) is said to be β-compact modulo an ideal or simply βI-compact if for every cover $\{U_\lambda : \lambda\in\Lambda\}$ of X by β-open sets of X, there exists a finite subset Λ_0 of Λ such that $X-\cup\{U_\lambda:\lambda\in\Lambda_0\}\in I$.

Theorem 3.2 For a space (X, τ, I) the following statements are equivalent:

(a) (X, τ, I) is βI-compact.

(b) (X, τ^*, I) is βI-compact.

(c) $(X, \beta O(X), I)$ is I-compact.

(d) For any family $\{F_\lambda:\lambda\in\Lambda\}$ of β-closed sets of X such that $\cap\{F_\lambda:\lambda\in\Lambda\}=\varnothing$, there exists a finite subset Λ_0 of Λ such that $\cap\{ F_\lambda:\lambda\in\Lambda_0\}\in I$.

Proof (a) □ (b) Let $\{U_\lambda : \lambda\in\Lambda\}$ be a τ^*-β-open cover of X such that $U_\lambda=V_\lambda - E_\lambda$, where $V_\lambda\in\beta O(X, \tau)$ and $E_\lambda\in I$. Now $\{V_\lambda : \lambda\in\Lambda\}$ is β-open cover of X and so there exists a finite subset Λ_0 of Λ such that $X - \cup\{V_\lambda : \lambda\in\Lambda_0\}=E\in I$. This implies that $X - \cup\{U_\lambda:\lambda\in\Lambda_0\}\subseteq E\cup[\cup\{E_\lambda : \lambda\in\Lambda_0\}]\in I$. Therefore (X, τ^*, I) is βI-compact.

(b) □ (a) Follows directly from the fact that $\tau\subseteq\tau^*$.

(a) □ (c) It is obvious.

(a) □ (d) Let $\{F_\lambda : \lambda\in\Lambda\}$ be a family of β-closed sets of X such that $\cap\{F_\lambda : \lambda\in\Lambda\}=\varnothing$. Then $\{X-F_\lambda : \lambda\in\Lambda\}$ is a β-open cover of X. By (a) there exists a finite subset Λ_0 of Λ such that $X-\cup\{X-F_\lambda:\lambda\in\Lambda_0\}$. This implies that $\cap\{F_\lambda : \lambda\in\Lambda_0\}\in I$.

(d) □ (a) Let $\{U_\lambda : \lambda\in\Lambda\}$ be a β-open cover of X, then $\{X-U_\lambda : \lambda\in\Lambda\}$ is a collection of β-closed sets and $\cap\{X-U_\lambda : \lambda\in\Lambda\}=\varnothing$. Hence there exists a finite subset Λ_0 of Λ such that $\cap\{X-U_\lambda : \lambda\in\Lambda_0\}\in I$. This implies that $X-\cup\{U_\lambda : \lambda\in\Lambda_0\}\in I$.Therefore (X, τ, I) is βI-compact.

The following Theorems are obvious and the proofs are thus omitted.

Theorem 3.3 For a space (X, τ, I) the following statements are equivalent:

(a) (X, τ) is β-compact.

(b) (X, τ, I_f) is βI_f-compact.

(c) $(X, \tau, \{\varnothing\})$ is $\beta\{\varnothing\}$-compact.

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Theorem 3.4 Let I and J be two ideals on a space (X,τ) with $I \subseteq J$. If (X,τ, I) is βI-compact, then it is βJ-compact.

Corollary 3.5 For any ideal I, every β-compact space is βI-compact.

Corollary 3.6 (a) If $(X, \tau , I\cap J)$ is β $(I\cap J)$-compact, then (X, τ, I) is βI-compact and (X, τ, J) is βJ-compact.
(b) If (X, τ, I) is βI-compact and (X, τ, J) is βJ-compact, then $(X, \tau, I\vee J)$ is $\beta(I\vee J)$-compact.

If (X,τ) is submaximal and E.D. and from the fact $\tau=SO(X, \tau)=\beta O(X, \tau)=\gamma O(X, \tau)$, we deduce the following theorem.

Theorem 3.7 If (X,τ) is submaximal and E.D., then each of βI-compactness, SI-compactness, γI-compactness and I-compactness are equivalent.

From the Definition of ideal I_c, the following theorem is obvious.

Theorem 3.8 If (X,τ, I_c) is βI_c-compact, then (X, τ) is β-Lindelof

Corollary 3.9 If (X,τ, I_c) is βI_c-compact, then (X, τ) is Lindelof.

Theorem 3.10 If $f:(X, \tau, I)\rightarrow(Y, \sigma)$ is a β-irresolute (resp., β-continuous) surjection and (X, τ, I) is βI-compact, then $(Y, \sigma, f(I))$ is βf(I)-compact (resp., f(I)-compact).

Proof Let $\{V_\lambda : \lambda\in\Lambda\}$ be a β-open (resp., open) cover of Y. Then $\{f^{-1}(V_\lambda) : \lambda\in\Lambda\}$ is a β-open cover of X and hence there exists a finite subset Λ_0 of Λ such that $X-\{f^{-1}(V_\lambda) : \lambda\in\Lambda_0\}\in I$. Since f is surjective, by Lemma 2.8. we have $Y- \cup\{V_\lambda : \lambda\in\Lambda_0\}=f(X - \cup\{f^{-1}(V_\lambda) : \lambda\in\Lambda_0\}\in f(I)$. Therefore, $(Y, \sigma, f(I))$ is βf(I)-compact (resp., f(I)-compact).

Theorem 3.11 If $f:(X, \tau)\rightarrow(Y, \sigma, J)$ is an M-β-open bijection and (Y, σ, J) is βJ-compact, then $(X, \tau, f^{-1}(J))$ is $\beta f^{-1}(J)$-compact.

Proof Since $f:(X, \tau)\rightarrow(Y, \sigma, J)$ is an M-β-open bijection, then f^{-1} is a β-irresolute surjection. Since (Y, σ, J) is βJ-compact, by Theorem 3.10., we obtain that $(X, \tau, f^{-1} (J))$ is $\beta f^{-1} (J)$-compact.

4 COUNTABLY βI-COMPACT SPACES

In this section, we introduce a class of countably compact spaces in terms of ideals called countably βI-compact.

Definition 4.1 A space (X, τ, I) is said to be countably βI-compact if for every countable β-open $\{U_n: n\in N\}$ of X there exists a finite subset N_0 of N such that $X-\cup\{U_n: n\in N_0\}\in I$, where N denotes the set of positive integers.

From the Definitions, we have the following relationships:

βI-compact □ countably β I-compact
↓ ↓
γI-compact □ countably γ I-compact
↓ ↓
SI-compact □ countably SI-compact
↓ ↓
I-compact □ countably I-compact

The reverse implication does not hold. However, we have the following result:

Theorem 4.2 If (X, τ, I) is countably βI-compact and (X, τ) is β-Lindelof space, then (X, τ, I) is βI-compact.

Proof Obvious.

Theorem 4.3 For a space (X, τ, I) the following statements are equivalent:
(a) (X, τ, I) is countably βI-compact.
(b) For any countable family $\{F_n : n\in N\}$ of β-closed sets of X such that $\cap\{F_n : n\in N\}=\varnothing$ there exists a finite subset N_0 of N such that $\cap\{F_n : n\in N_0\}\in I$.
(c) (X, τ^*, I) is countably βI-compact.
Proof (a) □ (b) Let $\{F_n : n\in N\}$ be a countable family of β-closed sets of X such that $\cap\{F_n : n\in N\}=\varnothing$. Then

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$\{X - F_n : n \in N\}$ is a countable β-open cover of X. By (a) there exists a finite subset N_0 of N such that $X - \cup\{X - F_n : n \in N_0\} \in I$. This implies that $\cap\{F_n : n \in N_0\} \in I$.
(b) □ (a) Let $\{U_n : n \in N\}$ be a countable β-open cover of X, then $\{X - U_n : n \in N\}$ is a countable collection of β-closed sets and $\cap\{X - U_n : n \in N\} = \varnothing$. Hence there exists a finite subset N_0 of N such that $\cap\{X - U_n : n \in N_0\} \in I$. Therefore, we have $X - \cup\{U_n : n \in N_0\} \in I$. This show that (X, τ, I) is countably βI-compact.
(a)□(c)Obvious.

The following two Theorems are easily obtained and the proofs are thus omitted.

Theorem 4.4 Let (X, τ, I) be countably βI-compact. If J is an ideal on X with $I \subseteq J$, then (X,τ, J) is countably βJ -compact.

Theorem 4.5 For a space (X, τ) the following statements are equivalent:
(a) (X,τ) is countably β-compact.
(b) (X,τ, I_f) is countably βI_f -compact.
(c) (X,τ, {∅}) is countably β{∅}-compact.

Corollary 4.6 (a) If $(X, \tau, I \cap J)$ is countably $\beta(I \cap J)$-compact, then (X,τ, I) is countably βI-compact and (X, τ, J) is countably βJ -compact.
(b) If (X, τ, I) is countably βI-compact and (X, τ, J) is countably βJ -compact, then $(X, \tau, I \vee J)$ is countably $\beta(I \vee J)$-compact.

Theorem 4.7 Let $f:(X, \tau, I) \rightarrow (Y, \sigma)$ be an β-irresolute (resp., β-continuous) surjection function. If (X, τ, I) is countably βI-compact, then (Y, σ, f(I)) is countably βf(I)-compact (resp., countably f(I)-compact).

Proof Let $\{V_n : n \in N\}$ be a countable family of β-open (resp., open) cover of Y. Then $\{f^{-1}(V_n) : n \in N\}$ is a countable β-open cover of X and hence there exists a finite subset N_0 of N such that $X - \cup\{f^{-1}(V_n) : n \in N_0\} \in I$. Now since f is surjective, we have $Y - \cup\{V_n : n \in N_0\} = f(X - \cup\{f^{-1}(V_n) : n \in N_0\} \in f(I)$. Therefore, (Y, σ, f(I)) is countably βf(I)-compact (resp., countably f(I)-compact).

Theorem 4.8 If $f:(X, \tau) \rightarrow (Y, \sigma, J)$ is an M-β-open surjection and (Y, σ, J) is countably βJ-compact, then $(X, \tau, f^{-1}(J))$ is countably $\beta f^{-1}(J)$-compact.

Proof Since f is an M-β-open surjection and by Theorem 4.7., we obtain that $(X, \tau, f^{-1}(J))$ is countably $\beta f^{-1}(J)$-compact

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, β - open sets and β - continuous mappings, Bull. Fac. Sci. Assiut Univ.; 12, 1983, 77- 90.

[2] M. E. Abd El-Monsef, A. M. Kozae a and R. A. Abu-Gdairi, generalized near rough probability in topological spaces, Int. J. Contemp. Math. Sciences, Vol. 6, No. 23, 2011, 1099 - 1110.

[3] M. E. Abd El-Monsef, E. F. lashien and A. A. Nasef, S-compactness via ideals, Qater Univ. Sci. J. Vol. 11, 1991, 33 - 44.

[4] D. Andrijevi, Semi-preopen sets, ibid. 38, 1986, 24 - 32.

[5] D. E. Cameron, A class of maximal topologies, pacific, J. Math. 70, 1977, 101 - 104.

[6] T. R. Hamlett and D. Jankovic, Compactness with respect to an ideal, Boll. U. M. .I. 7, 1990, no. 4-B, 849 - 861.

[7] K. Kuratowski, Topology, Vol. I, Academic press, New York, 1966.

[8] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70, 1963, 36 - 41.

[9] R. A. Mahmoud and M. E. Abd El-Monsef, β -irresolute and β -topological invariant, Proc. Pakistan Acad. Sci. 27, 1999, 285 - 296.

[10] A. A. Nasef, Some classes of compactness in terms of ideals, Soochow J. Math. 27 (3), 2001, 343 - 352.

[11] A. A. Nasef and T. Noiri, Strongly β-irresolute functions, Jour. of Natural Sci. and Math. V. 36, 1996, no. 2, 199 - 206.

[12] R. L. Newcomb, Topologies which are compact modulo an ideal, Ph. D. Dissertation, Univ. of Cal. at Santa Barbara, 1967.

[13] D. V. Rancin, Compactness modulo an ideal, Dokl Akad. Nauk SSSR 202, 1972, 193 - 197.

[14] R.Vaidyanathaswamy, Set Topology, 2nded.chelsea, 1960.

[15] S. Willard, General topology, Addison-wesley, readings, Mass., 1970.